

THE DERIVED CATEGORY OF SHEAVES OF COMMUTATIVE DG RINGS (PREVIEW)

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ABSTRACT. In this short paper we outline (mostly without proofs) our new approach to the derived category of sheaves of commutative DG rings. The proofs will appear in a subsequent paper.

Among other things, we explain how to form the derived intersection of two closed subschemes inside a given algebraic scheme X , without recourse to simplicial or higher homotopical methods, and without any global assumptions on X .

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0. INTRODUCTION

In the theory of *DG schemes* – a simplified variant of *derived algebraic geometry* – it is important to have a suitable way to resolve a sheaf of rings by a *flat sheaf of DG rings*. A typical problem is this: X is a scheme, and $Y_1, Y_2 \subseteq X$ are closed subschemes. The *derived intersection* of Y_1 and Y_2 is a DG scheme

$$(Y, \mathcal{O}_Y) = Y_1 \times_X^{\mathbf{R}} Y_2$$

whose underlying topological space is $Y = Y_1 \cap Y_2$, and the structure sheaf

$$\mathcal{O}_Y = \mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{Y_2}$$

is a suitable sheaf of commutative DG rings on this space. If there exist flat resolutions $\phi_i : \mathcal{A}_i \rightarrow \mathcal{O}_{Y_i}$, by which we mean that \mathcal{A}_i is a flat commutative DG \mathcal{O}_X -ring, and ϕ_i is a DG ring quasi-isomorphism, then we can take

$$\mathcal{O}_Y := (\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2)|_Y.$$

(Of course, it is enough to resolve only one of the tensor factors.)

Date: 15 August 2016.

In case X is an affine scheme, or it is quasi-projective over a nice base ring \mathbb{K} , then it is quite easy to produce flat quasi-coherent DG ring resolutions of \mathcal{O}_{Y_i} , and in this way to construct the sheaf of DG rings \mathcal{O}_Y . This was already done in the paper [CK] of Ciocan-Fontanine and Kapranov.

But in general (for an arbitrary scheme X) there does not seem to be an existing method to obtain flat DG ring resolutions as sheaves on X itself. Thus the derived intersection (Y, \mathcal{O}_Y) has until now existed only as an object in a much more complicated homotopical setting. See the preprint [Be] of Behrend for one approach, and a survey of the approaches of Toën et al. and of Lurie under “derived stack” in [nLab]. Being only an object of a complicated homotopy category, the derived intersection (Y, \mathcal{O}_Y) is usually quite difficult to manipulate geometrically, and to form associated structures, such a derived module category over \mathcal{O}_Y , etc.

The first main innovation in this paper is the use of *commutative pseudo-semi-free sheaves of DG rings*. These sheaves enable the formation of flat commutative DG \mathcal{O}_X -ring resolutions in great generality. But before saying what these sheaves are, we must present a few background concepts.

For an open set $U \subseteq X$, let us denote by $\mathcal{O}_{U \subseteq X}$ the extension by zero to X of the sheaf \mathcal{O}_U . Suppose $I = \coprod_{n \leq 0} I^n$ is a graded set, and for each $i \in I$ we are given an open set $U_i \subseteq X$. Define

$$\mathcal{E}^n := \bigoplus_{i \in I^n} \mathcal{O}_{U_i \subseteq X}$$

and

$$\mathcal{E} := \bigoplus_{n \leq 0} \mathcal{E}^n.$$

We call \mathcal{E} the *pseudo-free graded \mathcal{O}_X -module* indexed by I . The summand \mathcal{E}^n is in degree n . Warning: the \mathcal{O}_X -module \mathcal{E} is usually not quasi-coherent!

The commutative tensor ring of \mathcal{E} over \mathcal{O}_X is called a *commutative pseudo-free graded ring*, and we denote it by $\mathcal{O}_X[I]$. See Section 1 for details. It is useful to view $\mathcal{O}_X[I]$ as a commutative pseudo-polynomial graded \mathcal{O}_X -ring, in which the elements $t_i := 1 \in \Gamma(U_i, \mathcal{O}_X)$ play the role of variables (and we call them pseudo-generators). For each point $x \in X$ the stalk $\mathcal{O}_X[I]_x$ is a genuine commutative polynomial graded $\mathcal{O}_{X,x}$ -ring, in variables indexed by the graded set $\{i \in I \mid x \in U_i\}$.

A sheaf of commutative DG \mathcal{O}_X -rings $\tilde{\mathcal{A}}$ is called pseudo-semi-free if the graded sheaf of rings $\tilde{\mathcal{A}}^{\natural}$, that is gotten by forgetting the differential on $\tilde{\mathcal{A}}$, is a commutative pseudo-free graded \mathcal{O}_X -ring. As a DG \mathcal{O}_X -module, such $\tilde{\mathcal{A}}$ is K-flat.

Any sheaf of commutative DG \mathcal{O}_X -rings \mathcal{A} admits a commutative pseudo-semi-free DG \mathcal{O}_X -ring resolution $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$. This is Theorem 2.7. Furthermore, these resolutions are unique up to suitable homotopies, that we explain below.

This brings us to the second main innovation of this paper. Consider the category $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$ of commutative DG \mathcal{O}_X -rings. We introduce a relation that we call *relative quasi-homotopy* on the set of morphisms in this category. See Definition 3.2. This is a congruence, and hence there is the *homotopy category*, in the genuine sense, that we denote by $\mathrm{K}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X)$. Its objects are the same as those of $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$, and its morphisms are the relative quasi-homotopy classes.

We can also form the abstract localization of $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$ with respect to the quasi-isomorphisms, and the result is the *derived category of commutative DG \mathcal{O}_X -rings*, that we denote by $\mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X)$. There is a commutative diagram of functors

$$(0.1) \quad \begin{array}{ccc} \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X & & \\ \downarrow \mathrm{P} & \searrow \mathrm{Q} & \\ \mathrm{K}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X) & \xrightarrow{\bar{\mathrm{Q}}} & \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X) \end{array}$$

The functor $\bar{\mathrm{Q}}$ is a *right Ore localization* with respect to the set of quasi-isomorphisms, and it is also *faithful*. This gives us very tight control on the morphisms in the derived category.

The commutative pseudo-semi-free DG rings have a certain lifting property that makes everything work. See Theorem 2.11. However, the commutative pseudo-semi-free DG rings have a built-in finiteness property (basically coming from the fact that only a finite intersection of open sets of X is open), thus preventing them from being “cofibrant objects”. This seems to indicate that there is no Quillen model structure on $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$.

We can now say how we solve the problem of derived intersection. For $i = 1, 2$ we view \mathcal{O}_{Y_i} as living in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$, and we choose pseudo-semi-free resolutions $\mathcal{A}_i \rightarrow \mathcal{O}_{Y_i}$. Define the topological space $Y := Y_1 \cap Y_2 \subseteq X$ and the commutative DG \mathcal{O}_X -ring

$$\mathcal{O}_Y := (\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2)|_Y.$$

Then the derived intersection of (Y_1, \mathcal{O}_{Y_1}) and (Y_2, \mathcal{O}_{Y_2}) is the DG ringed space (Y, \mathcal{O}_Y) . There is a canonical isomorphism

$$\mathcal{O}_Y \cong \mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X}^{\mathrm{L}} \mathcal{O}_{Y_2}$$

in $\mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X)$. See Corollary 5.2 for details. Furthermore, on any affine open set $V \subseteq X$ there is a canonical isomorphism in $\mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_V)$ between $\mathcal{O}_Y|_V$ and any quasi-coherent DG sheaf presentation of the derived intersection. This is explained in Sections 4 and 5 of the paper.

When the scheme X is quasi-projective, it is not hard to see that our approach is compatible with that of [CK]. For more general X we did not attempt a comparison, but it is almost certain that our approach is compatible with those of Behrend, Toën and Lurie.

It stands to reason that the method outlined in this paper should allow a clean construction of the *cotangent complex* of X ; see Remark 5.3. Our method should also permit a geometric version of Shaul’s *derived completion of DG rings* from [Sh], but we do not have a formulation of it yet.

Presumably our work can be extended without too much difficulty to sites that are more general than the Zariski topology of a scheme (e.g. to algebraic spaces, and maybe even to algebraic stacks). We leave this exploration aside for the time being.

In the body of the paper (before Section 5) we do not work with schemes, but rather with *topological spaces* in general. Indeed, the natural geometric object to which our approach applies is a *commutative DG ringed space*, which is a pair (X, \mathcal{A}) , consisting of a topological space X and a sheaf of commutative DG rings \mathcal{A} on it. Moreover, there is no need for a special base ring, such as a field of characteristic 0; our constructions are valid over \mathbb{Z} .

The present paper is only a preview, meant to convey our new ideas on this subject. Only a few proofs are given here (and some of them are just partial proofs). Our most important result is Theorem 2.7 on the existence of pseudo-semi-free resolutions, and for that we provide a sketch of a proof (the beginning of a full proof, and an indication how to complete it). Understanding the geometric principle of the proof of Theorem 2.7, and combining it with the proofs of several algebraic results in [Ye2], should presumably allow experts to write their own proofs of the rest of the theorems in this paper. At any rate, we intend to publish a complete account of our approach in the future. Until then, we welcome feedback from readers, with suggestions of proofs, of better results, and also of counterexamples and refutations, in case there should be any...

Acknowledgments. I wish to thank Liran Shaul, Rishy Vyas, Vladimir Hinich and Donald Stanley for discussions.

1. PSEUDO-FREE SHEAVES OF COMMUTATIVE GRADED RINGS

Let us fix a nonzero commutative base ring \mathbb{K} . For instance, \mathbb{K} could be a field of characteristic 0; or it could be the ring of integers \mathbb{Z} . Let X be a topological space. The constant sheaf on X with values in \mathbb{K} is \mathbb{K}_X . The category of \mathbb{K}_X -modules (i.e. sheaves of \mathbb{K} -modules on X) is $M(\mathbb{K}_X) = \text{Mod } \mathbb{K}_X$.

Definition 1.1. A *sheaf of commutative graded \mathbb{K}_X -rings* is a sheaf of graded rings $\mathcal{A} = \bigoplus_{m \leq 0} \mathcal{A}^m$, together with a homomorphism $\mathbb{K}_X \rightarrow \mathcal{A}^0$, that \mathcal{A} has the strong commutativity property: any local sections $a \in \mathcal{A}^m$ and $b \in \mathcal{A}^n$ satisfy $b \cdot a = (-1)^{mn} \cdot a \cdot b$, and $a \cdot a = 0$ if m is odd.

The category of sheaves of commutative graded \mathbb{K}_X -rings is denoted by $\text{GR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$.

Suppose $U \subseteq X$ is an open set, with inclusion morphism $g : U \rightarrow X$. For any \mathbb{K}_U -module \mathcal{N} its extension by zero to X is the \mathbb{K}_X -module $g_!(\mathcal{N})$; and for any \mathbb{K}_X -module \mathcal{M} its restriction to U is the \mathbb{K}_U -module $g^{-1}(\mathcal{M}) = \mathcal{M}|_U$. These operations are adjoint: there is a canonical isomorphism

$$(1.2) \quad \text{Hom}_{\mathbb{K}_X}(g_!(\mathcal{N}), \mathcal{M}) \cong \text{Hom}_{\mathbb{K}_U}(\mathcal{N}, g^{-1}(\mathcal{M}))$$

in $M(\mathbb{K})$. See [Ha, Section II.1].

Definition 1.3. Let $U \subseteq X$ be an open set, with inclusion morphism $g : U \rightarrow X$. The *pseudo-free \mathbb{K}_X -module of pseudo-rank 1 and pseudo-support U* is the \mathbb{K}_X -module

$$\mathbb{K}_{U \subseteq X} := g_!(\mathbb{K}_U).$$

The element

$$t_U := 1 \in \Gamma(U, \mathbb{K}_{U \subseteq X}) \cong \Gamma(U, \mathbb{K}_X)$$

is called the *pseudo-free generator* of the \mathbb{K}_X -module $\mathbb{K}_{U \subseteq X}$.

Taking $\mathcal{N} = \mathbb{K}_U$ in formula (1.2), we have canonical isomorphisms

$$(1.4) \quad \mathrm{Hom}_{\mathbb{K}_X}(\mathbb{K}_{U \subseteq X}, \mathcal{M}) \cong \mathrm{Hom}_{\mathbb{K}_U}(\mathbb{K}_U, g^{-1}(\mathcal{M})) \cong \Gamma(U, \mathcal{M})$$

in $M(\mathbb{K})$. The pseudo-free generator t_U can be interpreted as a homomorphism $t_U : \mathbb{K}_{U \subseteq X} \rightarrow \mathbb{K}_X$ in $M(\mathbb{K}_X)$. This is actually an injective homomorphism, and thus we can view $\mathbb{K}_{U \subseteq X}$ as an *ideal sheaf* in \mathbb{K}_X . It is the ideal sheaf “pseudo-generated” by t_U .

For a point $x \in X$ we denote by $\mathbb{K}_{U \subseteq X, x}$ the stalk of the sheaf $\mathbb{K}_{U \subseteq X}$ at x . If $x \in U$, then the stalk $\mathbb{K}_{U \subseteq X, x}$ is a free \mathbb{K} -module of rank 1 with basis t_U . But if $x \notin U$ then $\mathbb{K}_{U \subseteq X, x} = 0$. If $U' \subseteq X$ is another open set, then

$$\mathbb{K}_{U \subseteq X} \otimes_{\mathbb{K}_X} \mathbb{K}_{U' \subseteq X} \cong \mathbb{K}_{U \cap U' \subseteq X}$$

canonically as \mathbb{K}_X -modules. The pseudo-free generators multiply:

$$t_U \otimes t_{U'} \mapsto t_{U \cap U'}.$$

The pseudo-free sheaves were used to great effect by Grothendieck in [RD, Section II.7]. See also our paper [Ye1, Section 3], where the name “pseudo-free” was first used.

Definition 1.5. A *generator specification* on X is a triple

$$(I, \{U_i\}_{i \in I}, \{n_i\}_{i \in I})$$

consisting of a set I , a collection $\{U_i\}_{i \in I}$ of open sets of X , and a collection $\{n_i\}_{i \in I}$ of nonpositive integers n_i .

We often refer to the generator specification just as I , leaving the rest of the ingredients implicit. The numbers n_i are called the cohomological degrees. For every index i there is the pseudo-free generator

$$(1.6) \quad t_i := t_{U_i} \in \Gamma(U_i, \mathbb{K}_{U_i \subseteq X}).$$

Suppose a generator specification I is given. For any n let

$$(1.7) \quad I^n := \{i \in I \mid n_i = n\}.$$

Thus $I = \coprod_n I_n$, so it is a graded set. For any $x \in X$ we let

$$(1.8) \quad I_x := \{i \in I \mid x \in U_i\}.$$

Definition 1.9. Given a generator specification I , the *graded pseudo-free \mathbb{K}_X -module pseudo-generated by I* is

$$\mathcal{E} := \bigoplus_{n \leq 0} \mathcal{E}^n,$$

where for every n the graded component of cohomological degree n is

$$\mathcal{E}^n := \bigoplus_{i \in I^n} \mathbb{K}_{U_i \subseteq X}.$$

Note that for any point $x \in X$ the stalk \mathcal{E}_x is a graded free \mathbb{K} -module, with basis indexed by the graded set I_x .

Definition 1.10. The *noncommutative pseudo-free graded \mathbb{K}_X -ring* pseudo-generated by I is

$$\mathbb{K}_X \langle I \rangle := \bigoplus_{l \geq 0} \mathcal{E} \otimes_{\mathbb{K}_X} \cdots \otimes_{\mathbb{K}_X} \mathcal{E},$$

where in the l -th summand there are l tensor factors. The multiplication is the tensor product.

Note that $\mathbb{K}_X \langle I \rangle$ is actually bigraded: it has the tensor grading l and the cohomological grading n ; but we are only interested in the cohomological grading.

Definition 1.11. Let I be a generator specification. The *commutative pseudo-free graded \mathbb{K}_X -ring* pseudo-generated by I is the quotient $\mathbb{K}_X[I]$ of the \mathbb{K}_X -ring $\mathbb{K}_X \langle I \rangle$, modulo the two-sided ideal sheaf pseudo-generated by the local sections

$$t_i \cdot t_j - (-1)^{n_i \cdot n_j} \cdot t_j \cdot t_i$$

for all $i, j \in I$, and by the local sections $t_i \cdot t_i$ for all i such that n_i is odd.

The homogeneous component of $\mathbb{K}_X[I]$ of cohomological degree n is denoted by $\mathbb{K}_X[I]^n$. Thus

$$(1.12) \quad \mathbb{K}_X[I] = \bigoplus_{n \leq 0} \mathbb{K}_X[I]^n.$$

Proposition 1.13. For any point $x \in X$ there is a canonical graded \mathbb{K} -ring isomorphism

$$\mathbb{K}_X[I]_x \cong \mathbb{K}[I_x],$$

where $\mathbb{K}[I_x]$ is the commutative graded polynomial ring on the collection of graded variables indexed by the graded set I_x .

Corollary 1.14. For each n the sheaf $\mathbb{K}_X[I]^n$ is flat over \mathbb{K}_X .

Proposition 1.15. Let I be a generator specification, let $\mathcal{A} \in \text{GR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$, and for every $i \in I$ let $a_i \in \Gamma(U_i, \mathcal{A}^{n_i})$. Then there is a unique homomorphism $\phi : \mathbb{K}_X[I] \rightarrow \mathcal{A}$ in $\text{GR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$ such that $\phi(t_i) = a_i$.

Proof. Let \mathcal{E} be the pseudo-free \mathbb{K}_X -module pseudo-generated by I . The adjunction property (1.4) says that there is a unique homomorphism $\phi : \mathcal{E} \rightarrow \mathcal{A}$ in $\text{M}(\mathbb{K}_X)$ such that $\phi(t_i) = a_i$ on U_i . This extends uniquely to a homomorphism of \mathbb{K}_X -rings $\phi : \mathbb{K}_X \langle I \rangle \rightarrow \mathcal{A}$ by the universal property of the tensor product. To be explicit, any pseudo-monomial

$$t_{i_1} \otimes \cdots \otimes t_{i_l} \in \Gamma(U_{i_1} \cap \cdots \cap U_{i_l}, \mathbb{K}_X \langle I \rangle)$$

goes to the element

$$a_{i_1} \cdots a_{i_l} \in \Gamma(U_{i_1} \cap \cdots \cap U_{i_l}, \mathcal{A}).$$

Because \mathcal{A} is commutative, the two-sided ideal of graded commutators goes to zero, and therefore there is an induced homomorphism $\phi : \mathbb{K}_X[I] \rightarrow \mathcal{A}$. The uniqueness is clear. \square

Proposition 1.16. *Let I be a generator specification, let $\mathcal{A} \in \mathrm{GR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, and define $\mathcal{B} := \mathcal{A} \otimes_{\mathbb{K}_X} \mathbb{K}_X[I]$. Suppose for every $i \in I$ we are given an element $b_i \in \Gamma(U_i, \mathcal{B}^{n_i+1})$.*

- (1) *There is a unique derivation $d : \mathcal{B} \rightarrow \mathcal{B}$ of degree +1 that extends the differential of \mathcal{A} and such that $d(t_i) = b_i$.*
- (2) *If $d(b_i) = 0$ for all i , then $d \circ d = 0$.*

Proof. Like the previous proof, combined with [Ye2, Lemma 3.20]. \square

2. PSEUDO-SEMI-FREE SHEAVES OF COMMUTATIVE DG RINGS

Again X is a topological space.

Definition 2.1. A commutative DG \mathbb{K}_X -ring is a sheaf of commutative graded \mathbb{K}_X -rings $\mathcal{A} = \bigoplus_{i \leq 0} \mathcal{A}^i$, as in Definition 1.1, with a \mathbb{K}_X -linear differential d of degree +1 that satisfies the graded Leibniz rule.

A homomorphism of DG \mathbb{K}_X -rings $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of sheaves that respects the DG \mathbb{K}_X -ring structure. The category of commutative DG \mathbb{K}_X -rings is denoted by $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$.

In more conventional language, a commutative DG \mathbb{K}_X -ring \mathcal{A} would be called a sheaf of unital associative commutative nonpositive cochain differential graded \mathbb{K} -algebras on X .

Commutative rings are viewed as DG rings concentrated in degree 0.

Definition 2.2. A commutative DG ringed space over \mathbb{K} is a pair (X, \mathcal{A}) , where X is a topological space, and \mathcal{A} commutative DG \mathbb{K}_X -ring.

Definition 2.3. Let (X, \mathcal{A}) be a commutative DG ringed space over \mathbb{K} . A commutative DG \mathcal{A} -ring is a pair (\mathcal{B}, ϕ) , where $\mathcal{B} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$. The morphisms between commutative DG \mathcal{A} -rings are the obvious ones. The resulting category is denoted by $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$.

To any $\mathcal{A} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$ we can assign its cohomology $H(\mathcal{A})$, which is a graded \mathbb{K}_X -ring. Note that $H(\mathcal{A})$ is the sheaf associated to the presheaf $U \mapsto H(\Gamma(U, \mathcal{A}))$. Cohomology is a functor

$$H : \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X \rightarrow \mathrm{GR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X.$$

A homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is called a *quasi-isomorphism* if $H(\phi)$ is an isomorphism.

Let \mathcal{A} be a commutative DG \mathbb{K}_X -ring. We denote by \mathcal{A}^\natural the graded \mathbb{K}_X -ring gotten by forgetting the differentials. The commutative pseudo-free graded \mathbb{K}_X -ring pseudo-generated by a generator specification I was introduced in Definition 1.11.

Definition 2.4. Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$. We say that ϕ is a *pseudo-semi-free DG ring homomorphism* in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, and that \mathcal{B} is a *commutative pseudo-semi-free DG \mathcal{A} -ring*, if there is an isomorphism

$$\mathcal{B}^\natural \cong \mathcal{A}^\natural \otimes_{\mathbb{K}_X} \mathbb{K}_X[I]$$

of graded \mathcal{A}^\natural -rings, for some generator specification I .

Proposition 2.5. *Let \mathcal{B} be a pseudo-semi-free commutative DG \mathcal{A} -ring. Then \mathcal{B} is K -flat as a DG \mathcal{A} -module.*

Definition 2.6. Let (X, \mathcal{A}) be a commutative DG ringed space, and let \mathcal{B} be a commutative DG \mathcal{A} -ring. A *pseudo-semi-free commutative DG ring resolution* of \mathcal{B} over \mathcal{A} , or a *pseudo-semi-free resolution of \mathcal{B} in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$* , is a pair $(\tilde{\mathcal{B}}, \psi)$, where $\tilde{\mathcal{B}}$ is a pseudo-semi-free commutative DG \mathcal{A} -ring, and $\psi : \tilde{\mathcal{B}} \rightarrow \mathcal{B}$ is a surjective quasi-isomorphism of DG \mathcal{A} -rings.

Here is the most important result of this paper.

Theorem 2.7. *Let (X, \mathcal{A}) be a commutative DG ringed space, and let \mathcal{B} be a commutative DG \mathcal{A} -ring. There exists a commutative pseudo-semi-free DG ring resolution of \mathcal{B} over \mathcal{A} .*

Sketch of Proof. This is a geometrization of the proof of [Ye2, Theorem 3.21(1)], replacing variables by pseudo-generators. Instead of [Ye2, Lemmas 3.19 and 3.20], here we use Propositions 1.15 and 1.16.

As in the proof of [Ye2, Theorem 3.21(1)], we shall construct an ascending sequence $F_0(\tilde{\mathcal{B}}) \subseteq F_1(\tilde{\mathcal{B}}) \subseteq \dots$ of pseudo-semi-free DG rings in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, together with a compatible sequence of homomorphisms $\phi_q : F_q(\tilde{\mathcal{B}}) \rightarrow \mathcal{B}$. Moreover, there will be an ascending sequence $F_0(I) \subseteq F_1(I) \subseteq \dots$ of generator specifications, and compatible isomorphisms

$$F_q(\tilde{\mathcal{B}})^\natural \cong \mathcal{A}^\natural \otimes_{\mathbb{K}_X} \mathbb{K}_X[F_q(I)]$$

of graded \mathcal{A}^\natural -rings. The following conditions will be satisfied:

- (i) The graded sheaf homomorphisms $\phi_q : F_q(\tilde{\mathcal{B}}) \rightarrow \mathcal{B}$, $B(\phi_q) : B(F_q(\tilde{\mathcal{B}})) \rightarrow B(\mathcal{B})$ and $H(\phi_q) : H(F_q(\tilde{\mathcal{B}})) \rightarrow H(\mathcal{B})$ are surjective in degrees $\geq -q$.
- (ii) The graded sheaf homomorphism $H(\phi_q) : H(F_q(\tilde{\mathcal{B}})) \rightarrow H(\mathcal{B})$ is bijective in degrees $\geq -q + 1$.

In condition (i), $B(-)$ denotes coboundaries. The DG ring

$$\tilde{\mathcal{B}} := \lim_{q \rightarrow} F_q(\tilde{\mathcal{B}})$$

and the homomorphism

$$\phi := \lim_{q \rightarrow} \phi_q : \tilde{\mathcal{B}} \rightarrow \mathcal{B}$$

will have the desired properties.

We shall just start the actual proof, for $q = 0$, imitating the proof of [Ye2, Theorem 3.21(1)], and emphasizing the geometric considerations that arise here.

For any point $x \in X$ we have the ring homomorphism $\mathcal{A}_x^0 \rightarrow \mathcal{B}_x^0$. Let $B^0(\mathcal{B}_x^0)$ be the module of 0-coboundaries. We choose a collection $\{c_k''\}_{k \in K_{0,x}''}$ of elements in \mathcal{B}_x^{-1} , indexed by a set $K_{0,x}''$, such that the collection $\{d(c_k'')\}_{k \in K_{0,x}''}$ generates $B^0(\mathcal{B}_x^0)$ as an \mathcal{A}_x^0 -module. Let $J_{0,x}''$ be another set, with a bijection $d : K_{0,x}'' \rightarrow J_{0,x}''$. Define

$b_j'' := d(c_k'') \in \mathcal{B}_x^0$ for any $k \in K_{0,x}''$ and $j = d(k) \in J_{0,x}''$, so we have a collection $\{b_j''\}_{j \in J_{0,x}''}$ of elements of \mathcal{B}_x^0 .

Next choose a collection $\{b_j'\}_{j \in J_{0,x}'}$ of elements in \mathcal{B}_x^0 , indexed by a set $J_{0,x}'$, such that the collection $\{b_j''\}_{j \in J_{0,x}''} \cup \{b_j'\}_{j \in J_{0,x}'}$ generates \mathcal{B}_x^0 as an \mathcal{A}_x^0 -ring.

The indexing sets $J_{0,x}'$, $J_{0,x}''$ and $K_{0,x}''$ here correspond, respectively, to the indexing sets Y_0' , Y_0'' and Z_0'' in the proof of [Ye2, Theorem 3.21(1)]. Indeed, they would be the same if $X = \{x\}$, a space with a single point in it.

For any index $k \in K_{0,x}''$ there is an open neighborhood U_k'' of x such the element $c_k'' \in \mathcal{B}_x^{-1}$ extends to an element $c_k'' \in \Gamma(U_k'', \mathcal{B}^{-1})$. This choice also gives us an element

$$b_j'' := d(c_k'') \in \Gamma(U_k'', \mathcal{B}^0)$$

for $j = d(k) \in J_{0,x}''$. Likewise, for any index $j \in J_{0,x}'$ there is an open neighborhood U_j' of x such the element $b_j' \in \mathcal{B}_x^0$ extends to an element $b_j' \in \Gamma(U_j', \mathcal{B}^0)$.

Define the set

$$F_0(I_x) := J_{0,x}' \sqcup J_{0,x}'' \sqcup K_{0,x}''.$$

For $i \in F_0(I_x)$ define the open set $U_i := U_k''$ if either $i = k \in K_{0,x}''$ or $i = d(k) \in J_{0,x}''$; and define $U_i := U_j'$ if $i = j \in J_{0,x}'$. Define the integer $n_i := -1$ if $i = k \in K_{0,x}''$; and define $n_i := 0$ if $i \in J_{0,x}''$ or $i \in J_{0,x}'$. Thus we have a generator specification

$$(2.8) \quad (F_0(I_x), \{U_i\}_{i \in F_0(I_x)}, \{n_i\}_{i \in F_0(I_x)})$$

“around x ”.

Taking the union of (2.8) over all points $x \in X$ we get a “global” generator specification

$$(F_0(I), \{U_i\}_{i \in F_0(I)}, \{n_i\}_{i \in F_0(I)}).$$

Define the DG ring

$$F_0(\tilde{\mathcal{B}}) := \mathcal{A} \otimes_{\mathbb{K}_X} \mathbb{K}_X[F_0(I)]$$

with differential $d(t_k) := t_{d(k)}$ for any index $k \in K_{0,x}'' \subseteq F_0(I_x) \subseteq F_0(I)$. This is possible by Proposition 1.16. According to Proposition 1.15 there is a homomorphism $\phi_0 : F_0(\tilde{\mathcal{B}}) \rightarrow \mathcal{B}$ of DG \mathcal{A} -rings. Checking at stalks we see that ϕ_0 satisfies condition (i) above for $q = 0$; condition (ii) is trivial for $q = 0$.

At this stage we can shrink the indexing set $F_0(I)$, as long as condition (i) holds for $q = 0$. See Remark 5.4 regarding the possibility to make the set $F_0(I)$ finite.

From here on the construction of $F_q(\tilde{\mathcal{B}})$ for $q \geq 1$ continues along the lines of the proof of [Ye2, Theorem 3.21(1)], with geometric arguments very similar to those we used above: for every q we go to stalks at points, choose elements, and extend them to open sets. \square

Definition 2.9. Let $\eta : \mathcal{A} \rightarrow \mathcal{A}^+$ be a homomorphism in $\text{DGR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$. We say that η is a *split acyclic pseudo-semi-free homomorphism* if η is pseudo-semi-free and a quasi-isomorphism, and there is a homomorphism $\epsilon : \mathcal{A}^+ \rightarrow \mathcal{A}$ in $\text{DGR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$ such that $\epsilon \circ \eta = \text{id}_{\mathcal{A}}$.

Theorem 2.10. Suppose $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a quasi-isomorphism in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$. Then ϕ can be factored as $\phi = \phi^+ \circ \eta$, where $\phi^+ : \mathcal{A}^+ \rightarrow \mathcal{B}$ is a surjective quasi-isomorphism, and $\eta : \mathcal{A} \rightarrow \mathcal{A}^+$ is a split acyclic pseudo-semi-free homomorphism.

In a commutative diagram:

$$\begin{array}{ccccc}
 & & \mathcal{A}^+ & & \\
 & \swarrow \epsilon & \uparrow \eta & \searrow \phi^+ & \\
 \mathcal{A} & \xleftarrow{\mathrm{id}} & \mathcal{A} & \xrightarrow{\phi} & \mathcal{B}
 \end{array}$$

Sketch of Proof. We actually prove more: there is a split contractible commutative pseudo-semi-free DG ring \mathcal{C} , and a homomorphism $\mathcal{C} \rightarrow \mathcal{B}$, such that $\mathcal{A}^+ = \mathcal{A} \otimes_{\mathbb{K}_X} \mathcal{C}$. \square

Theorem 2.11. Let $\mathcal{A} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}$, let $\mathcal{B} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, and for $i = 0, 1$ let $\phi_i : \tilde{\mathcal{B}}_i \rightarrow \mathcal{B}$ be quasi-isomorphisms in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$. Then there exists a pseudo-semi-free DG ring $\tilde{\mathcal{B}}' \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, together with quasi-isomorphisms $\psi_i : \tilde{\mathcal{B}}' \rightarrow \tilde{\mathcal{B}}_i$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, such that $\phi_0 \circ \psi_0 = \phi_1 \circ \psi_1$.

The statement is shown in the next commutative diagram in the category $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$.

$$\begin{array}{ccccc}
 & & \tilde{\mathcal{B}}' & & \\
 & \swarrow \psi_0 & & \searrow \psi_1 & \\
 \tilde{\mathcal{B}}_0 & & & & \tilde{\mathcal{B}}_1 \\
 & \searrow \phi_0 & & \swarrow \phi_1 & \\
 & & \mathcal{B} & &
 \end{array}$$

Sketch of Proof. This is similar to the proof of [Ye2, Theorem 3.22]. But the pseudo-semi-free DG ring $\tilde{\mathcal{B}}'$ has to be tailored, in terms of the open sets involved, to the DG rings $\tilde{\mathcal{B}}_0$ and $\tilde{\mathcal{B}}_1$. \square

Of course, by induction, this can be extended to any *finite* number of quasi-isomorphisms $\phi_i : \tilde{\mathcal{B}}_i \rightarrow \mathcal{B}$.

Remark 2.12. The construction of the DG ring $\tilde{\mathcal{B}}$ in Theorem 2.11 involves refinement. There is finiteness built into it (since open sets allow only finite intersections). In general, a single $\tilde{\mathcal{B}}'$ will not work for an infinite collection of quasi-isomorphisms $\phi_i : \tilde{\mathcal{B}}_i \rightarrow \mathcal{B}$.

This seems to indicate that there are no cofibrant objects in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, and thus there is no Quillen model structure!

3. RELATIVE QUASI-HOMOTOPIES AND THE DERIVED CATEGORY

The next definition is a variant of the left homotopy from Quillen theory (see [Ho]). The DG ring \mathcal{B}^+ plays the role of a *cylinder object*.

Definition 3.1. Let $\mathcal{A} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, and let $\phi_0, \phi_1 : \mathcal{B} \rightarrow \mathcal{C}$ be homomorphisms in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$. A *homotopy between ϕ_0 and ϕ_1 relative to \mathcal{A}* is a commutative diagram

$$\begin{array}{ccccc} \mathcal{B} & \xleftarrow{\mu} & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} & \xrightarrow{\phi_0 \otimes \phi_1} & \mathcal{C} \\ & \searrow \epsilon & \downarrow \eta & \nearrow \phi & \\ & & \mathcal{B}^+ & & \end{array}$$

in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, where μ is the multiplication homomorphism, and ϵ is a quasi-isomorphism. If a homotopy exists, then we say that ϕ_0 and ϕ_1 are *homotopic relative to \mathcal{A}* .

Definition 3.2. Let $\phi_0, \phi_1 : \mathcal{B} \rightarrow \mathcal{C}$ be homomorphisms in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$. The homomorphisms ϕ_0 and ϕ_1 are said to be *quasi-homotopic relative to \mathcal{A}* if there is a quasi-isomorphism $\psi : \tilde{\mathcal{B}} \rightarrow \mathcal{B}$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$ such that $\phi_0 \circ \psi$ and $\phi_1 \circ \psi$ are homotopic relative to \mathcal{A} , in the sense of Definition 3.1. This relation on morphisms in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$ is called *relative quasi-homotopy*.

$$\begin{array}{ccccc} & & \phi_i \circ \psi & & \\ & \searrow & \curvearrowright & \nearrow & \\ \tilde{\mathcal{B}} & \xrightarrow{\psi} & \mathcal{B} & \xrightarrow{\phi_i} & \mathcal{C} \end{array}$$

Theorem 3.3. Suppose $\tilde{\phi}_0, \tilde{\phi}_1 : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{C}}$, $\phi_0, \phi_1 : \tilde{\mathcal{B}} \rightarrow \mathcal{C}$ and $\sigma : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ are homomorphisms in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, such that that $\phi_i = \sigma \circ \tilde{\phi}_i$, σ is a quasi-isomorphism, and the homomorphisms ϕ_0 and ϕ_1 are homotopic relative to \mathcal{A} . Then there is a pseudo-semi-free resolution $\tilde{\psi} : \tilde{\mathcal{B}}' \rightarrow \tilde{\mathcal{B}}$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, such that $\tilde{\phi}_0 \circ \tilde{\psi}$ and $\tilde{\phi}_1 \circ \tilde{\psi}$ are homotopic relative to \mathcal{A} .

Here are the commutative diagrams in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, for $i = 0, 1$:

$$\begin{array}{ccccc} \tilde{\mathcal{B}}' & \xrightarrow{\tilde{\psi}} & \tilde{\mathcal{B}} & & \\ & \searrow \tilde{\phi}_i \circ \tilde{\psi} & \downarrow \tilde{\phi}_i & \searrow \phi_i & \\ & & \tilde{\mathcal{C}} & \xrightarrow{\sigma} & \mathcal{C} \end{array}$$

Theorem 3.4. Let $\mathcal{A} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$. The relation of relative quasi-homotopy is a congruence on the category $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$.

In order to have a visible distinction from the Quillen model category notation, below we choose notation that resembles the Grothendieck notation in [RD].

Definition 3.5. Let $\mathcal{A} \in \text{DGR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$. The *homotopy category* of $\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}$ is its quotient category modulo the relative quasi-homotopy congruence, and we denote it by $\text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$.

Thus for any pair of objects \mathcal{B}, \mathcal{C} we have

$$\text{Hom}_{\text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})}(\mathcal{B}, \mathcal{C}) = \frac{\text{Hom}_{\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}}(\mathcal{B}, \mathcal{C})}{\text{relative quasi-homotopy}}.$$

There is a functor

$$P : \text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A} \rightarrow \text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$$

that is the identity on objects and surjective on morphisms.

Within $\text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$ we have the set of quasi-isomorphisms, and they form a multiplicatively closed set of morphisms.

Theorem 3.6. *The quasi-isomorphisms in $\text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$ satisfy the right Ore condition and the right cancellation condition.*

Definition 3.7. Let $\mathcal{A} \in \text{DGR}_{\text{sc}}^{\leq 0}/\mathbb{K}_X$. The *derived category* of $\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}$ is its localization with respect to the quasi-isomorphisms. We denote it by $\text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$.

By definition there is a functor

$$Q : \text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A} \rightarrow \text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$$

that is the identity on objects. Since relatively quasi-homotopic morphisms in $\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}$ become equal in $\text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$, we get a commutative diagram of functors

$$\begin{array}{ccc} \text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A} & & \\ \downarrow P & \searrow Q & \\ \text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}) & \xrightarrow{\bar{Q}} & \text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}) \end{array}$$

Corollary 3.8. *The functor*

$$\bar{Q} : \text{K}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}) \rightarrow \text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$$

is a right Ore localization, and it is also faithful.

This tells us that any morphism in $\text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$ can be expressed as a simple right fraction:

$$Q(\phi) \circ Q(\psi)^{-1}$$

where ϕ, ψ are morphisms in $\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}$, and ψ is a quasi-isomorphism. Moreover, there is equality

$$Q(\phi_1) = Q(\phi_2)$$

in $\text{D}(\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A})$ iff ϕ_1 and ϕ_2 are relatively quasi-homotopic in $\text{DGR}_{\text{sc}}^{\leq 0}/\mathcal{A}$.

We do not wish to perform a detailed study of maps between commutative DG ringed spaces in this paper. We only note that:

Proposition 3.9. *Let $V \subseteq X$ be an open set. The restriction functor*

$$\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A} \rightarrow \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}|_V, \quad \mathcal{B} \mapsto \mathcal{B}|_V$$

induces functors

$$\mathrm{K}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}) \rightarrow \mathrm{K}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}|_V)$$

and

$$\mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}) \rightarrow \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}|_V),$$

that commute with the functors Q , P and $\bar{\mathrm{Q}}$.

4. LEFT DERIVED TENSOR PRODUCTS OF SHEAVES OF DG RINGS

As before, (X, \mathcal{A}) is a commutative DG ringed space over \mathbb{K} .

Theorem 4.1. *Consider the commutative DG ringed space (X, \mathcal{A}) . There is a bi-functor*

$$(- \otimes_{\mathcal{A}}^{\mathrm{L}} -) : \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}) \times \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}) \rightarrow \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}),$$

together with a morphism

$$\xi : \mathrm{Q} \circ (- \otimes_{\mathcal{A}} -) \rightarrow (- \otimes_{\mathcal{A}}^{\mathrm{L}} -)$$

of bifunctors

$$\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A} \times \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A} \rightarrow \mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}),$$

with this property: if $\mathcal{B}, \mathcal{C} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$ are such that at least one of them is K -flat over \mathcal{A} , then the morphism

$$\xi_{\mathcal{B}, \mathcal{C}} : \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C} \rightarrow \mathcal{B} \otimes_{\mathcal{A}}^{\mathrm{L}} \mathcal{C}$$

in $\mathrm{D}(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A})$ is an isomorphism.

Sketch of Proof. Given $\mathcal{B}, \mathcal{C} \in \mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$ we choose pseudo-semi-free resolutions $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ and $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{A}$, and define

$$\mathcal{B} \otimes_{\mathcal{A}}^{\mathrm{L}} \mathcal{C} := \tilde{\mathcal{B}} \otimes_{\mathcal{A}} \tilde{\mathcal{C}}.$$

The results of Section 3 shows that this is a derived functor. \square

The derived tensor product respects localizations, as in Proposition 3.9.

5. RESOLUTIONS IN ALGEBRAIC GEOMETRY

In this section (X, \mathcal{O}_X) is a scheme (over the base ring \mathbb{K}).

Let \mathcal{A} be a quasi-coherent commutative DG \mathcal{O}_X -ring; by this we mean that each \mathcal{O}_X -module \mathcal{A}^p is quasi-coherent. Let $V \subseteq X$ be an affine open set, and write $C := \Gamma(V, \mathcal{O}_X)$ and $A := \Gamma(V, \mathcal{A})$. So A is a commutative DG C -ring. We can build a commutative semi-free DG C -ring resolution $g : \tilde{A} \rightarrow A$. Each \tilde{A}^p is a C -module, and we can sheafify it to get a quasi-coherent sheaf $\tilde{\mathcal{A}}^p$ on V . In this way we obtain a commutative semi-free DG \mathcal{O}_V -ring resolution $g : \tilde{\mathcal{A}} \rightarrow \mathcal{A}|_V$.

Theorem 5.1. *Let (X, \mathcal{O}_X) be a scheme, let \mathcal{A} be a quasi-coherent commutative DG \mathcal{O}_X -ring, and let $V \subseteq X$ be an affine open set. Suppose we are given a commutative pseudo-semi-free DG \mathcal{O}_X -ring resolution $g : \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ on all of X , and also a commutative semi-free DG \mathcal{O}_V -ring resolution $g' : \tilde{\mathcal{A}}' \rightarrow \mathcal{A}|_V$ on V . Then there exists a commutative pseudo-semi-free DG \mathcal{O}_V -ring resolution $g'' : \tilde{\mathcal{A}}'' \rightarrow \mathcal{A}|_V$ on V , and DG \mathcal{O}_V -ring quasi-isomorphisms $f : \tilde{\mathcal{A}}'' \rightarrow \tilde{\mathcal{A}}|_V$ and $f' : \tilde{\mathcal{A}}'' \rightarrow \tilde{\mathcal{A}}'$, such that $g|_V \circ f = g' \circ f' = g''$.*

Proof. The commutative semi-free DG \mathcal{O}_V -ring resolution $g'' : \tilde{\mathcal{A}}'' \rightarrow \mathcal{A}|_V$ is just a special case of a commutative pseudo-semi-free DG \mathcal{O}_V -ring resolution; so we can apply Theorem 2.11 to it and to $g|_V : \tilde{\mathcal{A}}|_V \rightarrow \mathcal{A}|_V$. \square

What Theorem 5.1 says is that locally our commutative pseudo-semi-free resolutions are the same as the quasi-coherent resolutions that were considered in [CK].

In the next corollary we identify a sheaf on a closed subset $Y \subseteq X$ with its pushforward to X .

Corollary 5.2. *Let (Y_1, \mathcal{O}_{Y_1}) and (Y_2, \mathcal{O}_{Y_2}) be closed subschemes of (X, \mathcal{O}_X) . There is a commutative DG ringed space (Y, \mathcal{O}_Y) , such that*

$$Y = Y_1 \cap Y_2 \subseteq X$$

as topological spaces, and

$$\mathcal{O}_Y = \mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{Y_2}$$

in $D(\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X)$. Thus

$$(Y, \mathcal{O}_Y) = (Y_1, \mathcal{O}_{Y_1}) \times_{(X, \mathcal{O}_X)}^{\mathbf{R}} (Y_2, \mathcal{O}_{Y_2}),$$

the derived intersection of these subschemes.

Proof. For $i = 1, 2$ we choose pseudo-semi-free resolutions $\mathcal{A}_i \rightarrow \mathcal{O}_{Y_i}$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$. Let $Y := Y_1 \cap Y_2$, and define

$$\mathcal{O}_Y := (\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2)|_Y,$$

the restriction of the DG \mathcal{O}_X -ring $\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2$ to the closed subset Y . A calculation in stalks shows that the canonical DG ring homomorphism

$$\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2 \rightarrow \mathcal{O}_Y$$

is a quasi-isomorphism. \square

Remark 5.3. Here is a speculation regarding the *cotangent complex* of the scheme X . For this we view \mathcal{O}_X as living in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$, where \mathbb{K} is the base ring. Let $\mathcal{A} \rightarrow \mathcal{O}_X$ be a commutative pseudo-semi-free resolution in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathbb{K}_X$. There is a DG \mathcal{A} -module $\Omega_{\mathcal{A}/\mathbb{K}}^1$, defined as the sheafification of the presheaf

$$V \mapsto \Omega_{\Gamma(V, \mathcal{A})/\mathbb{K}}^1.$$

Let

$$\mathbf{L}_X := \mathcal{O}_X \otimes_{\mathcal{A}} \Omega_{\mathcal{A}/\mathbb{K}}^1 \in D(\mathcal{O}_X)$$

We believe that L_X is canonically isomorphic (in the derived category $D(\mathcal{O}_X)$) to the cotangent complex as constructed in [II].

Remark 5.4. If the scheme X is noetherian, and if \mathcal{A} is a coherent commutative \mathcal{O}_X -ring (e.g. $\mathcal{A} = \mathcal{O}_Y$ for a closed subscheme $Y \subseteq X$), then it is possible to find a commutative pseudo-semi-free resolution $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ in $\mathrm{DGR}_{\mathrm{sc}}^{\leq 0}/\mathcal{O}_X$ such that $\tilde{\mathcal{A}}^{\natural} \cong \mathcal{O}_X \otimes_{\mathbb{K}_X} \mathbb{K}_X[I]$, and the indexing set I is finite in each degree. This is by the results of [RD, Section II.7].

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